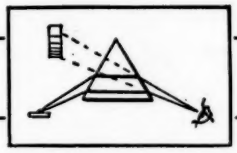
Journal of the OPTICAL SOCIETY *of* AMERICA

VOLUME 26
No. 9



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Published monthly
at Prince and Lemon Sts., Lancaster, Pa.
by the
AMERICAN INSTITUTE OF PHYSICS
Incorporated
175 Fifth Avenue, New York, N. Y.
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Journal of the
OPTICAL SOCIETY
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VOLUME 26 SEPTEMBER, 1936 NUMBER 9

The Calculation of Filters for Color Temperature Conversion

K. S. WEAVER, *Kodak Research Laboratories,* Rochester, N. Y.*

(Received June 30, 1936)

FILTERS for the conversion of the color temperature of a source of radiant energy from one value to another are of considerable importance in the testing and practical application of photographic materials and in the photometry of light sources of different color temperatures. For this purpose, glass, gelatin, and liquid filters of various kinds are used. The chart illustrated in this paper is useful in solving several problems involved in the design and examination of these filters.

Dr. H. P. Gage[1] has shown that, for a filter capable of converting an energy distribution from one color temperature to another, the density varies linearly with the reciprocal of the wave-length. This relation is derived from Wein's equation and is, therefore, subject to the same limitations as to accuracy at high temperatures or long wave-lengths.

Wein's equation for a temperature, T, is

$$J_{\lambda T}=c_1\lambda^{-5}e^{-c_2/\lambda T}. \tag{1}$$

The ratio of this expression for two temperatures, T_1 and T_2, is

$$J_{T\lambda_2}/J_{T\lambda_1}=e^{-(c_2/\lambda)(1/T_1-1/T_2)}. \tag{2}$$

This expression, therefore, gives as a function of wave-length, the transmission of a filter which will convert both in distribution and intensity, energy from a black radiator at a temperature, T_1, to that of a black radiator at a temperature, T_2.

It is evident that all such filters for which the expression $(1/T_1-1/T_2)$ is a constant will be identical without restriction as to the absolute values of the temperatures. The expression $(1/T_1-1/T_2)$ has been designated as "A" in previous work and this designation will be used, at times, in the present discussion.

The density of a filter is defined as the logarithm of the reciprocal of the transmission. Expression (2) written in logarithmic form becomes therefore

$$D_\lambda=(c_2/\lambda)(1/T_2-1/T_1)\log e \tag{3}$$

or

$$D_\lambda=-(c_2\log e/\lambda)A, \tag{4}$$

i.e., the density is a linear function of the reciprocal wave-length.

In Fig. 1, this function is shown graphically for both positive and negative values of A.

The scale of ordinates is in density units and that of abscissae is in reciprocal wave-length units. Since, according to (3) or (4), D is always equal to zero when λ is infinite, all of the straight lines representing ideal black-body conversion filters intersect at $\lambda=\infty$, $D=0$.

Straight lines which lie above the reciprocal wave-length axis represent yellow filters which

* Communication No. 597.

[1] H. P. Gage, "Color Filters for Altering Color Temperature. Pyrometer Absorption Daylight Glasses," J. O. S. A. **23**, 46 (1933).

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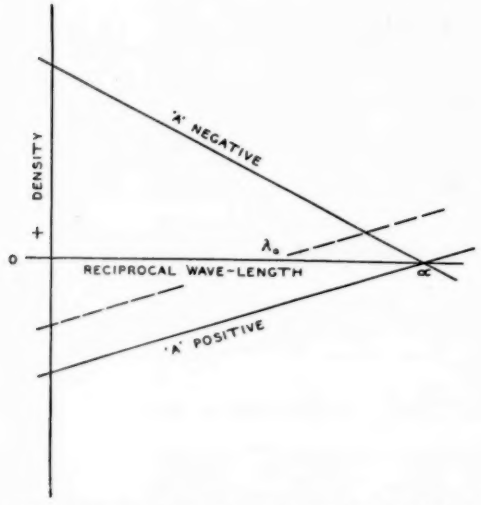

FIG. 1. The density reciprocal wave-length relation is linear for black-body conversion filters. A is positive for yellow filters and negative for blue filters. For blue filters the D *vs.* $1/\lambda$ line must be raised, intersecting the $D=0$ axis at $\lambda_0 \neq \infty$ in order to represent a real filter over a definite wave-length interval.

will convert black-body energy both in distribution and intensity from T_1 to T_2. These lines represent real filters which may actually be made to follow the given curve over a more or less extended wave-length interval. These filters are represented by the pyrometer absorbing glasses described by H. P. Gage.[1] On the other hand, although lines lying below the reciprocal wave-length axis represent blue filters which would convert black-body energy both in distribution and intensity from T_1 to T_2 they can have no real existence since the densities are all negative. In this case, by raising the line a certain amount without changing its slope, we may obtain a locus representing the distribution of transmitted energy of the hypothetical filter. This is generally sufficient for practical purposes.

Such a filter is represented by the dotted line in Fig. 1 and its equation may be derived by subtracting D_{λ_0} from (4),

$$D_\lambda' = D_\lambda - D_{\lambda_0},$$
$$= c_2 \log eA(1/\lambda_0 - 1/\lambda). \qquad (5)$$

It is, therefore, seen that the additional term $1/\lambda_0$ is always needed for a blue filter. It may or may not be used in the case of yellow filters since, when A is negative, either Eq. (4) or (5) represents a real filter within a limited wave-length interval. Eq. (4) may be considered identical with (5) for the special case when $\lambda_0 = \infty$.

The chart shown in Fig. 2 is a form used in this laboratory in determining the ideal characteristics of filters for particular conversions, or in estimating the quality of a given filter from its spectrophotometric curve. This chart corresponds to that portion of Fig. 1 covering the wave-length interval of the visible spectrum, since this region is generally of primary interest. In cases where the energy in other wave-length regions is of interest, similar charts may be prepared.

In Fig. 2 there is added a reciprocal temperature scale as shown; the density scale is placed on the right. The scale of reciprocal temperature is used in determining the slope of the line representing a filter covering a certain temperature step.

The scales of D, $1/T$, and $1/\lambda$ have been chosen arbitrarily as dictated by convenience. Since A and D in Eq. (5) for a particular filter are directly proportional, we may say that A is equal to D considered as a linear distance in the direction of ordinates for a certain reciprocal wave-length interval $(1/\lambda_1 - 1/\lambda_2)$. The scales of $1/T$ and D were so chosen that the ratio D/A for equal distances in the direction of ordinates has a value of 5000°K and λ_1 was made 380 mμ from which by Eq. (5) λ_2 is found to be 547 mμ. In Fig. 2 this is shown as an index line drawn at 547 mμ.

In using the chart, a straight line is drawn from the 380 mμ ordinate at T_1 to a point on the index line at 547 mμ opposite T_2 and extended to cut the axis of abscissae. This line may be raised or lowered arbitrarily to suit any special conditions or for comparison with any known filter without change in the color temperature conversion represented.

Such a line representing a blue filter converting from 2360°K to 2850°K is shown in Fig. 2. This filter is obviously denser than it need be for the color step given. Blue filters are available in glass, dyed gelatin, and in the form of liquids which have transmissions nearly, if not quite, equal to those of the ideal, although the densities at certain wave-lengths depart noticeably from those indicated by theory.

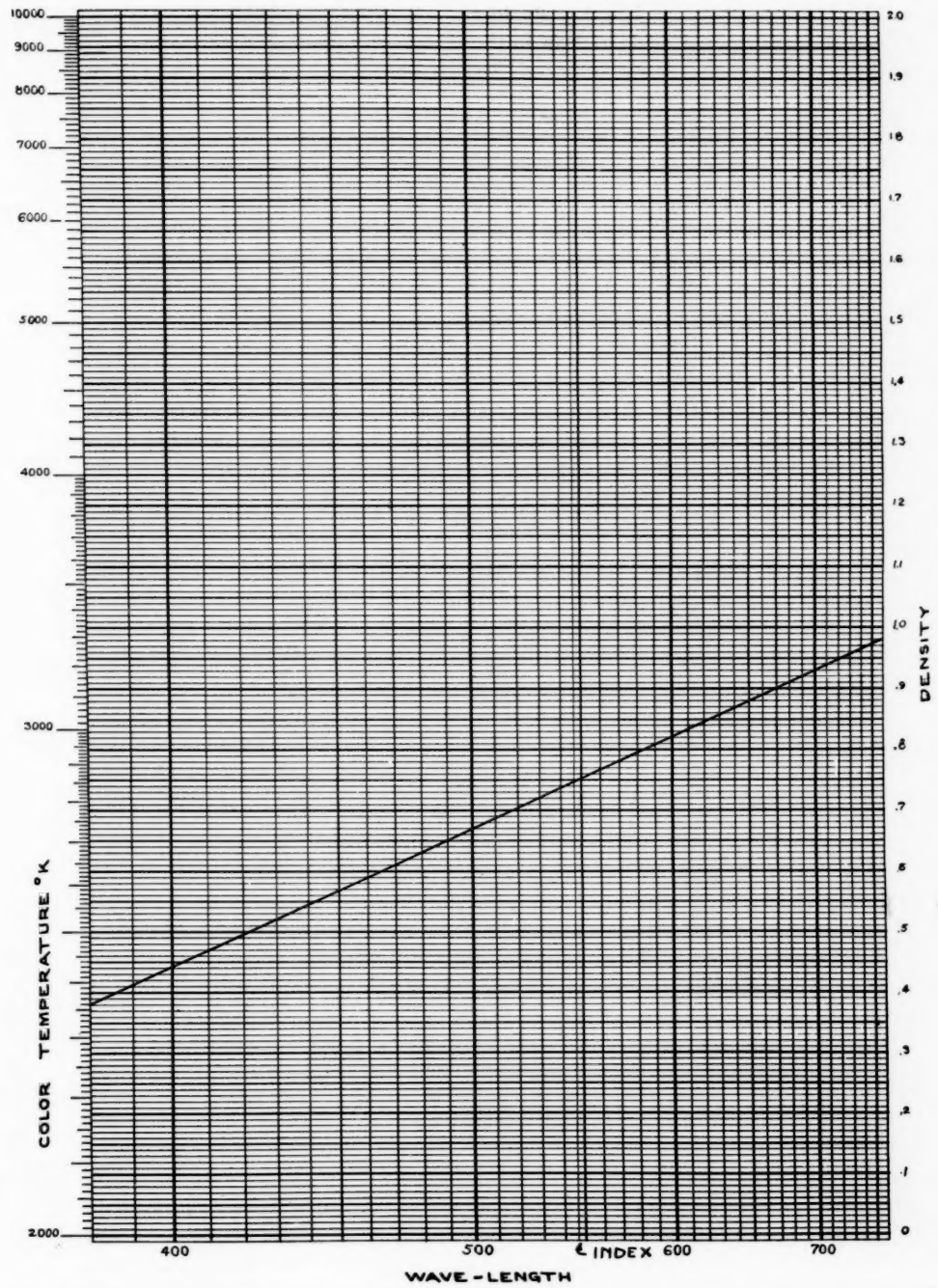

FIG. 2. Color temperature filter chart. The straight line representing a blue filter converting from 2360°K to 2850°K is drawn as an illustration. At temperatures above those for which Wein's equation is adequate, the ideal curve will depart slightly from a straight line.

Since in this chart the wave-length scale does not extend to infinity, if we wish to draw in the curve for $\lambda_0 = \infty$, we do it by solving Eq. (4) for two widely separated values of λ and drawing a straight line between these two points on the chart.

Theoretical Transmission of an Ideal Black-Body Conversion Filter

As already stated, a filter corresponding to Eq. (4) will transmit energy from a black-body radiator at T_1 equal in intensity and distribution to that from a black-body radiator T_2, where $(1/T_1 - 1/T_2) = A$.

The integral luminous transmission of such a filter will, therefore, equal the ratio of the brightness of a black body at T_2 to that of a black body at T_1. This may be written $t = \beta T_2/\beta T_1$, in which t is the integral luminous transmission of the filter for the whole visible spectrum and βT_1 and βT_2 are the brightnesses of a black body at temperatures T_1 and T_2, respectively.

The relation between βT and T over the interval 2000°K to 6000°K is well expressed by

$$\log \beta = 7.023 - 11{,}544/T + 844{,}200/T^2. \quad (6)$$

The constants of this formula were calculated by the use of the black-body energy data contained in the *International Critical Tables* together with the I.E.S. visibility data. Using the density notation, we may define $d' = -\log t$ and

$$d' = \log \beta T_1/\beta T_2.$$

By substitution from (6),

$$d' = 11{,}544(1/T_2 - 1/T_1) + 844{,}200(1/T_1^2 - 1/T_2^2). \quad (7)$$

If we change the value of λ_0 from ∞, for which the preceding expression applies, to another value in order to represent, in the case of a blue filter, one which does not involve negative densities, or in the case of a yellow filter, one which has a higher transmission, we must then subtract from (7) the density at the chosen value of λ_0 given by (4) which leads to the more general formula,

$$d = ((c_2 \log e)/\lambda_0 - 11{,}544)A + 844{,}200\ (1/T_1^2 - 1/T_2^2). \quad (8)$$

This expression is of interest because it enables one to calculate, without step-by-step integrations, the integral density or transmission of an ideal black-body conversion filter, for comparison with any specified filter.

The variation in integral luminous density with T_1 is given by the second term of Eq. (8). Real filters which approximate the ideal fairly well would probably show about the same variation in density with T_1.

In Fig. 3, a graphical representation of d as a function of A and T_1 is shown. This was calculated by the use of Eq. (8).

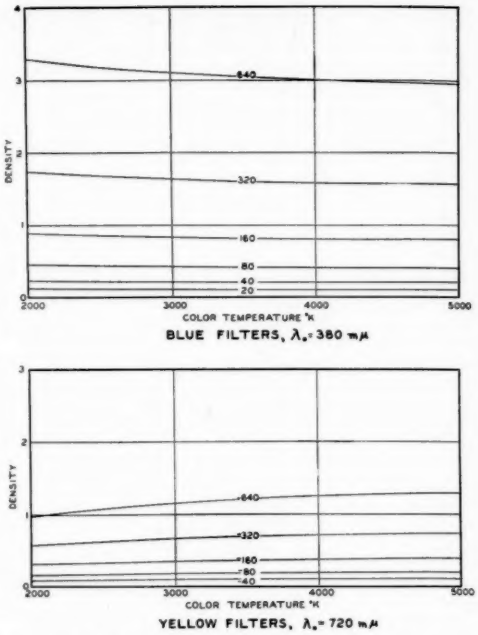

FIG. 3. Relation between integral density of an ideal black-body conversion filter and the conversion power, A, as calculated by the use of Eq. (8). The numbers on each curve give the value of A in microreciprocal degrees.

The Diffraction Grating at Grazing Incidence

LUIS W. ALVAREZ, *Ryerson Physical Laboratory, University of Chicago*

(Received March 5, 1936)

An experimental test of the grating law has been made by using a coarse, plane grating at grazing incidence, in a spectrograph similar to that used in the x-ray case. No deviations from the grating law were found which were as large as the experimental error of 0.02 percent.

INTRODUCTION

ALTHOUGH the relative values of x-ray wave-lengths are known with considerable accuracy, there has been a good deal of debate upon the subject of their absolute magnitudes. The tabulated wave-lengths are calculated from crystal diffraction experiments, on the assumption that the crystals are perfect, and that the electronic charge has the value 4.77×10^{-10} e.s.u. Bearden[1] has shown that the wave-lengths, as determined from ruled grating x-ray spectra, are uniformly 0.2 percent higher than the crystal values. Since the grating space of the crystal depends upon e[1], it was difficult to believe that the electronic charge could be in error by 0.6 percent, the amount needed to bring the wave-lengths into coincidence. The suggestion that the crystal planes near the surface might have a greater spacing than those in the interior was investigated experimentally, with negative results, and the possibility that there might be some unsuspected error in the grating formula when applied to the case of grazing incidence was considered theoretically by Eckart,[2] who showed that the usual equations should be valid in this case.

It must be remembered, however, that the grating law has not been thoroughly investigated from the experimental standpoint. Most present day optical wave-lengths are measured from grating spectra, but the grating formula is not used. The wave-lengths are merely determined by comparison with known standards, which have in turn been measured by interferometric means. The last precision measurement of optical wave-lengths in terms of grating space and diffraction angles was made in 1888 by Bell,[3] who used four of Rowland's best plane gratings. It is interesting to note that the value be published for the wave-length of the D_1 line of sodium is higher than the interferometer value by ten times the probable error of his experiments. Since this discrepancy is in the same direction as that of the x-ray case, and since he used angles near 45°, it seemed worth while to investigate the possibility that there might be some systematic error in grating measurements which would become more noticeable as the angle of incidence was increased. (While the work has been in progress, however, several papers have appeared which would make this latter view difficult to entertain.)[4, 5]

METHOD

As an experimental test of the possibility mentioned above, it was decided to measure the wave-length of a line in the optical region by means of the technique perfected by Bearden.[1] In order that the suspected systematic errors should be the same in this work with optical radiation as in Bearden's work with x-radiation, one might suppose that all linear dimensions of the apparatus ought to be increased in the same ratio as the wave-lengths. Such an increase was prohibited by practical considerations, but the various parts of the apparatus designed by Bearden were duplicated on an enlarged scale, even though the factor was necessarily different for different parts. A diagram of the apparatus is shown in Fig. 1. The wave-length and grating space were increased by a factor of 2400, the slits by 100, and the source to plate distance by about 10. The source of light ($\lambda=4358$A) was a water cooled mercury arc operating at atmospheric pressure. Each part of the long spectro-

[1] J. A. Bearden, Phys. Rev. **37**, 1210 (1931).
[2] C. Eckart, Phys. Rev. **44**, 12 (1933).
[3] L. Bell, Phil. Mag. **25**, 350 (1888).
[4] M. Söderman, Nature **135**, 67 (1935).
[5] G. Kellström, Nature **136**, 682 (1935).

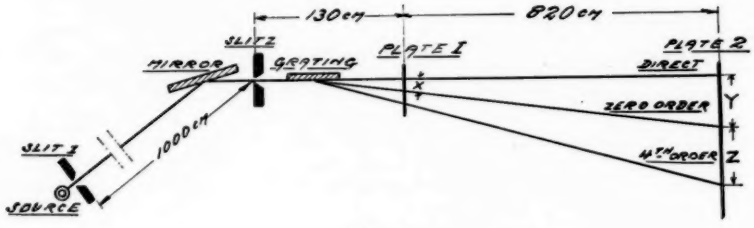

FIG. 1.

graph was mounted on a pillar of granite blocks. The grating was ruled on a seasoned surface plate casting, which was later polished flat to within less than one wave-length. The grooves were milled accurately, by setting with a microscope on a standard scale carried on the milling machine carriage. The width of the grooves was one-half the grating space.

The various pieces of apparatus were aligned on the same horizontal plane to within 0.1 cm with the aid of a surveyor's level, and adjusted vertically to within about 20 seconds of arc, by a modified Gauss eyepiece technique. It was most important to have the second slit accurately parallel to the grating surface, and this was accomplished by rotating the grating until the Lloyd type fringes had their maximum intensity. These Lloyd fringes were caused by scattered light interfering with the relatively weak light diffracted into the higher orders. They introduced an asymmetry into the diffracted line shapes, but were eliminated in the final sets of plates by a shield which screened off the scattered light.

MEASUREMENTS

The distance between the plateholders was measured with a surveyor's tape, which was later calibrated under the same tension against a standard meter. The grating space was compared directly with a standard decimeter scale, using two microscopes. The lines on the second plates were from 5 to 10 mm wide, so their centers could not be located without the use of a microphotometer. A fiducial mark was placed near the center of each line, and the distances between these marks measured on a comparator. Then each line was microphotometered, and its center of gravity located. By knowing the magnification ratio of the instrument, it was possible to determine the small distance between the fiducial mark and the center of the line. This value was then used as a correction to the comparator measured distance, to give the actual distance between the lines.

A fiducial line was ruled on either side of the diffracted line, with a razor. The reference marks for the direct and reflected beams were impressed on the plate during the exposure, and were the shadows of fine wires stretched vertically across the plateholder in such a way as to bisect the broad lines. The center of a line was found directly from its microphotometer trace. The lines were quite symmetrical, so it was merely necessary to mark the mean abscissa on the trace at several different ordinates; the points lay on a vertical line defining the center of the wide line. The midpoint of the fiducial mark, or pair of marks, was found in the same manner.

EXPERIMENTAL PROCEDURE

Before each series of plates was exposed, the adjustment of the slits, mirror, and grating was rechecked. The method of taking exposures was similar to that used by Bearden. The grating was moved but once, so that no errors could occur from resetting. "Creeping" of the grating carriage along its ways was eliminated by allowing it to stand fifteen minutes after it had been moved. Errors were introduced into the first set of measurements by thermal "boiling" of the images on the plate. This trouble was avoided by lengthening the exposures and guarding against air currents in the room. The shape of the diffracted lines was found to vary greatly with changes in the width of the second slit, angle of diffraction, and order. Under some conditions, the lines appeared as widely spaced doublets or triplets, but they could be focused by adjusting

the width of the grating slit. A small effect of this kind has been noted with optical gratings before.[6] The line shapes could be predicted accurately by considering the images of the two slits in the grating, as seen in the nth order of diffraction. In the zeroth order, of course, these image slits were the same distance from the grating as the actual slits, but in higher orders, their distances varied with the angles and the order. The line shapes were the calculated Fresnel diffraction patterns from a single slit, and could be obtained with the aid of a Cornu spiral. In the final set of plates, the slit width was adjusted to give the sharpest possible shape of the 4th-order blue line.

Calculations

The usual grating law $n\lambda = d(\sin i + \sin i')$ may be written more conveniently for the purpose of the experiment in the form:

$$n\lambda = 2d \sin [(2\theta+\alpha)/2] \sin (\alpha/2),$$

where θ is the glancing angle of incidence, and α the angle between the reflected and diffracted beams. Stauss and Porter have shown that there is a third order correction to this formula, which takes into account the horizontal divergence of the beam collimated by the slits. The complete equation becomes:

$$n\lambda = 2d\{\sin [(2\theta+\alpha)/2] \sin (\alpha/2)\} \times\{1+[3a^2/20(\phi^2-\theta^2)](\phi^2/l_2^2-\theta^2/l_1^2)\},$$

where a is the length of the illuminated portion of the grating, l_1 the distance between the grating and the source, and l_2 the distance between grating and image.

Calculation shows that this correction is negligible for the case of the second plateholder, but that it is appreciable for the first. The only diffracted line recorded on the first plate is the zero order, and for this the Stauss-Porter correction reduces to the form:

$$x = x'\{1+[3a^2/40](1/l_1^2-1/l_2^2)\},$$

where x is the measured distance between the direct and reflected lines, and x' is the value to be used in its place in the calculations. This factor affects the calculated wave-length by 0.01 or 0.02 percent, depending upon the position of the first plateholder.

[6] D. C. Stockbarger and L. Burns, J. O. S. A. **23**, 379 (1933).

TABLE I. *Sample calculation, Plate 90, correction terms from microphotometer traces in cm.*

	Trace Distance	Mean T.D.	Plate Distance
Direct Beam	−0.97, −0.94, −0.96	−0.96	−0.0192
Zero Order	−1.20, −1.32, −1.17	−1.23	−0.0246
4th Order Blue	+0.29	+0.29	+0.0058
	Comparator Setting	**Correction**	**Corrected Setting**
Direct Beam	0.0505	−0.0192	0.0313
Zero Order	20.9150	−0.0246	20.8904
4th Order	39.5269		
4th Order	39.7495		
4th Order, Mean	39.6382	+0.0058	39.6440

cm	cm	
20.8904	2.2559	
0.0313	2.2570	
	2.2572	$\theta = 0°\ 38'\ 52.39''$
$y = 20.8591$	$x = 2.2567$	$\alpha = 1°\ 9'\ 49.99''$
39.6440		$d = 4.0000$ cm
20.8904		$b = 822.30$ cm
		$\lambda = 4358.82$A
$z = 18.7536$		

The angles θ and α were calculated from the measured distances x, y, z, and b, by the use of the formulae:

$$\theta = \tfrac{1}{2} \sin^{-1} [y/(y^2+(yb/y-x)^2)^{\frac{1}{2}}]$$
$$\alpha = \sin^{-1} [z \cos 2\theta/\{(y+z)^2+(yb/y-x)^2\}^{\frac{1}{2}}].$$

Andoyer's 15 place trigonometric tables were used throughout, and all computations were made on a machine (see Table I).

Results

For the final measurements, two sets of five plates were exposed. In the first set, the grating slit was 1.06 mm wide, and the first holder was about 50 cm from the grating. In the second set, the holder was moved to a point one meter from the grating, and the slit width was increased to 1.38 mm. The lines on the plates of the second group were sharper, since there was less diffraction from the slit, and the correction term was not so sensitive to errors in the determination of the length of illumination on the grating. This length was calculated from the geometry of the apparatus, and also measured on a plate which had been exposed while fastened to the surface of the grating.

The values for the wave-length of the blue line were:

4362.93	4359.84
59.86	57.00
60.28	57.49
57.43	57.68
58.43	58.78

Because of the differences in the experimental arrangement, the second five plates were given twice as much weight as the first. The weighted mean of the ten determinations is then 4358.71 $\pm$0.35A. This probable error is from considerations of internal consistency alone. To calculate the actual probable error, we must include the errors in the measurement of b and d. b was known to better than 0.1 cm, and as it enters as a square into the calculation, the limiting error in the wave-length from this source is 0.02 percent. The graph of grating space showed a variation from one part of the surface to another; the spacing near the center was smaller than that near the edges. It was possible to locate the portion of the grating used in the taking of a particular plate, and the grating space at this section was then employed in the calculation of the wave-length from that plate. This could introduce an error into the determination, of the order of 0.01 percent. We may then conclude that the wave-length of the blue mercury line, as measured in this way is 4358.71 $\pm$0.8A. The accepted value, as given in Kayser's tables, is 4358.30A. This agreement upholds the view that the diffraction law is valid at grazing incidence. Further evidence that the grating law is accurate at glancing angles may be inferred from the fact that associated predictable phenomena, such as the Stauss-Porter correction and the variation of the line shapes, have been verified.

In conclusion, the author wishes to thank Professors H. G. Gale, A. H. Compton, and Carl Eckart for many helpful discussions of various aspects of the problem.

The Photographic Intermittency Effect in the Region of Extremely Short Flash Periods*

J. H. WEBB, *Eastman Kodak Research Laboratories*

(Received June 27, 1936)

Experiments were carried out to test the equivalence of intermittent exposures of the same average intensity for cases in which the flash period varies over the range 10^{-3} to 10^{-7} second. The exposures were made by means of a mirror mounted on top of a high speed air turbine. The results are in agreement with those obtained in former investigations made at longer flash times to the effect that above the fusion frequency of flash, intermittent exposures of the same average intensity are independent of the speed of flash.

IN a former work by the author[1] some measurements were carried out on the photographic intermittency effect covering the range of flash times from approximately 1 second to 0.0002 second. This work was concerned principally with investigating the relationship existing between the reciprocity law failure and intermittency effect. The results obtained in this work confirmed the work of previous investigators[2] who had found that a continuous exposure and an intermittent exposure of the same average intensity become identical at sufficiently high frequencies of flash and, in addition, showed that the fusion frequency for which the two types of exposure become identical varies directly with the average intensity of illumination. On the basis of these findings, an hypothesis was put forward at that time to correlate this behavior with the quantum nature of the light producing the exposure. In substance, this hypothesis ran as follows: From the quantum point of view a so-called continuous exposure of a photographic grain is actually intermittent; therefore, if, in a macroscopically intermittent exposure, the frequency of flash is increased to, or beyond, that point for which a grain receives on the average approximately one quantum of light during each flash, then the intermittent and continuous exposures become mathematically equivalent, and, therefore, should become experimentally indistinguishable to that grain. Obviously, if this condition is satisfied for the largest grains in the emulsion, then it is simultaneously satisfied for all smaller grains. On the basis of this hypothesis, which has considerable experimental[1] and theoretical[3] support, it may be seen that the condition which makes for indistinguishability of intermittent and continuous exposures would stand only to be improved by increasing the frequency of flash above the fusion frequency. Thus, once the fusion frequency of flash has been reached, there should be no further change of the photographic effect with varying frequency of flash or, alternatively, with decrease in the flash period. Recently, results have appeared in the literature,[4] which seem to throw some doubt on the validity of the above hypothesis at very short

* Communication No. 596 from the Kodak Research Laboratories.

[1] J. H. Webb, "Relationship between Reciprocity Law Failure and the Intermittency Effect in Photographic Exposure," J. O. S. A. **23**, 157–69 (1933).

[2] E. A. Baker, "Validity of Talbot's Law for the Photographic Plate," Proc. Opt. Soc. Convention (London), Part I, 238–44 (1926). H. Bouasse, *Vision et Réproduction des Formes et des Couleurs* (Librairie Delagrave, Paris, 1917, 456–7. A. E. Weber, "Über die Anwendung des rotierenden Sektors zur photographischen Photometrie," Ann. d. Physik **45**, 801–38 (1914). H. E. Howe, "Modification of the Hilger Sector Photometer Method for Measuring Ultra-Violet Absorption and Its Application in the Case of Certain Derivatives of Fluoran," Phys. Rev. **8**, 674–88 (1916). B. O'Brien, "Photographic Spectral Energy Measurements with a Spiral Aperture Disc," Phys. Rev. **33**, 640 (1929). B. O'Brien and E. Dickerman O'Brien, "Intermittent Exposure in Photographic Spectrophotometry over Wide Intensity Ranges," Phys. Rev. **37**, 471 (1931). B. O'Brien and V. L. Parks: "Photographic Reciprocity and Intermittency Defects near the Long Wave-Length Limit of Plate Sensitivity," Phys. Rev. **41**, 387–8 (1932). C. E. Weinland, "Intermittency Effects in Photographic Exposure," J. O. S. A. **15**, 337–53 (1927). F. Twyman and F. Simeon, "The Logarithmic Wedge Sector and Its Use in Quantitative Spectrum Analysis," Trans. Opt. Soc. (London) **31**, 169–83 (1929–30). K. S. Gibson, H. J. McNicholas, E. P. Tyndall, M. K. Frehafer and W. E. Mathewson, "The Spectral Transmissive Properties of Dyes," Bur. Standards Sci. Papers **18**, 121–84 (1923); F. Eckert and R. Pummerer, "Photographische Spektralphotometrie der Absorptionspektren von Farbstoffen," Zeits. f. physik. Chemie **87**, 599–618 (1914).

[3] L. Silberstein and J. H. Webb, "Photographic Intermittency Effect and the Discrete Structure of Light," Phil. Mag. **18**, 1–24 (1934).

[4] L. I. Zimmerman, "Time Lag in the Formation of the Latent (Photographic) Image," J. O. S. A. **23**, 342–52 (1933).

flash periods. The present experimental work was accordingly initiated to test this hypothesis in the region of extremely short flash times.

The experiments to be described here were carried out with a high speed revolving mirror by means of which an image of an illuminated slit was moved past a photographic plate. By making the slit narrow, the light arm long, and the speed of the rotating mirror high, it is possible to obtain by this method flashes of very short duration. The energy of an exposure given intermittently in this manner, and providing the over-all time of the exposure is long compared with the time of a single flash, is independent of the speed of the mirror. This is immediately obvious upon noting that, as the speed of the mirror is increased, the length of the separate flash periods is decreased in an amount sufficient exactly to compensate for the increased number of flashes. Accordingly, varying the frequency of flash affords a good means for giving a series of equal energy intermittent exposures with varying length flash periods. The comparison exposures in these experiments were of equal over-all time and were made to the same light source. All exposures of a single series were placed on the same film. Under these conditions, any variation of the photographic effect obtained is to be attributed to failure of the photographic material to integrate the exposure properly.

Apparatus

The arrangement of the apparatus for making the exposures in these experiments is shown in the schematic diagram of Fig. 1. As a light source, S, a tungsten vertical ribbon filament (18-ampere, 6-volt) lamp was used, placed in a metal housing containing a condenser lens, L_1, mounted in a tube projecting from one side. The filament of the lamp was imaged by means of the condenser lens on a vertical slit. As this slit was considerably smaller than the filament image itself, only the central, uniformly illuminated part of the filament image was used. The slit was imaged in turn by the long focal length lens, L_2, which was placed at a distance of 310 cm from the slit. The approximately parallel light emerging from this lens, after passing through a diaphragm, D, fell on the plane mirror, M, which was mounted on top of a high speed rotor. The light, after reflection, formed an image of the slit at a distance of 395 cm from the mirror and this image could be moved in the arc of a circle of this radius by rotating the mirror. The photographic plate, or film, upon which the exposures were made was placed in a holder, P, mounted at the rear end of a diaphragmed box which had a shutter at K for timing the exposures.

The high speed air turbine used for rotating the mirror is shown in Fig. 2. The rotor and stator assembly is an exact duplicate of a model lent to the author by Professor J. W. Beams. In this connection, it should be mentioned that the turbine performed beautifully throughout all the experiments, being completely stable and free from vibration over the range of speeds 150 to 1675 r.p.s. The line air pressure was 80 lbs. per sq. in. and the speed of the rotor was varied by means of a fine adjustment needle valve.

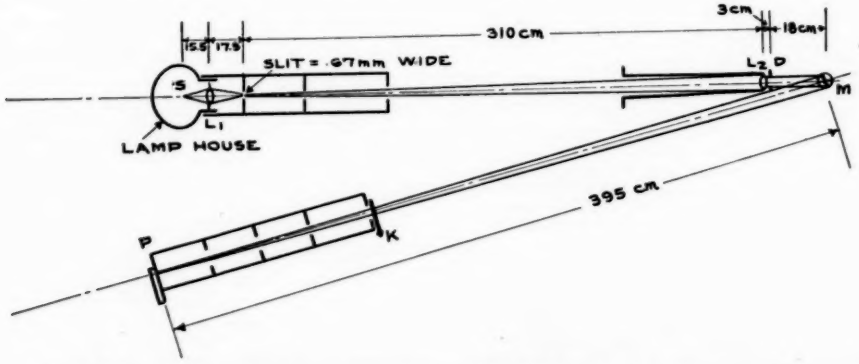

Fig. 1. Schematic diagram of apparatus used in making high speed intermittent exposures.

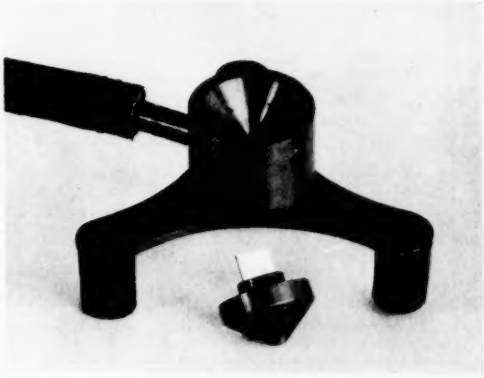

FIG. 2. High speed air turbine with mirror.

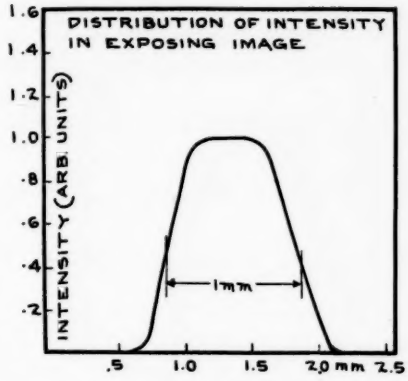

FIG. 3. Curves of intensity distribution in the exposing image used in high speed intermittent exposures.

The mirror was of stellite (1 mm thick) with only one polished face, and it possessed a clear aperture of 8 by 11 mm. The mirror was made up specially for this work by the Bausch and Lomb Optical Company. Care had to be taken in mounting the mirror not to distort the mirror optically, and at the same time to fasten the mirror securely enough to withstand the high rotational speeds. This was accomplished by inserting one edge of the mirror into a snugly fitting slot cut in the projecting top of the rotor, around which a steel ring was pressed to keep the mirror from moving laterally. The image formed by the optical system with the mirror in place presented a sharply defined, uncolored appearance when viewed on a piece of ground glass at the exposure plane. The very low aperture of the system (approximately $f : 200$) reduced errors from chromatic and spherical aberrations to a negligible amount.

The intensity distribution in the exposing image was obtained from a microdensitometer tracing of a photograph of the stationary image. By means of an intensity scale H and D curve for the photographic material used in making this photograph, the microdensitometer tracing was transformed into terms of intensity distribution in the image as shown in Fig. 3. The lack of sharpness of the image indicated by this curve is due mainly to the spread of light caused by reflection from the grains of the photographic emulsion. Therefore, in computing the length of the flash periods, the width of image was not taken as the full width at the base of this curve but as the width at an intermediate level, which gave the value 1.0 mm. This value was chosen by calculating the image width theoretically from the known slit width, using the given object and image distances. In this way the image width was found to be 0.9 mm, so the image width of 1.0 mm was considered as being a fair value for use in the computation of the flash times.

The speed of the air turbine was measured by means of a photo-cell, amplifier, and loudspeaker arrangement by means of which an audible note corresponding to the revolution frequency of the rotor was obtained. Using the method of beats this note was compared with the note emitted by a second loudspeaker connected to a calibrated oscillator circuit. In this way, the speed of the rotor could be measured with an accuracy of one revolution per second. It was found possible to hold the rotor speed constant to two or three revolutions per second by slight adjustments of the needle valve controlling the air pressure. The line pressure was very steady and it was seldom necessary to change the adjustment more than once or twice during an exposure of several minutes' duration.

EXPERIMENTAL PROCEDURE

The general procedure in these experiments consisted in making a series of comparison exposures at various fixed speeds of the rotor. A series of speeds which was found to cover the range of flash times of interest was: 180, 500,

1000, 1500, and 1675 r.p.s., the maximum speed being fixed by the maximum air pressure available. This series of speeds gave the corresponding series of flash times 11.2, 4.02, 2.01, 1.34, and 1.2×10^{-8} second, respectively, as calculated by the equation,

$$\Delta T=\Delta W/4\cdot\pi R\cdot N,$$

in which ΔW is the image width, R is the distance from mirror to the photographic plate, and N is the number of revolutions per second of the mirror. This series of flash times was used in making exposures on each of the photographic materials tested in these experiments.

After adjusting the speed of the rotor to the desired value, the exposure was started by opening the manually operated shutter K (Fig. 1). At the end of a specified time, which varied from eight to twenty minutes, depending upon the sensitivity of the emulsion being tested, the exposure was terminated by the same means. During the exposure the lamp intensity was held constant by maintaining the current in the lamp at its rated value of 18 amperes. The speed of the rotor was checked constantly during the exposure by comparing the note emitted by the rotor loudspeaker with that of the oscillator loudspeaker. Upon completion of the first exposure, the speed of the rotor was then set at a new value and another exposure made in similar manner.

All exposures of one series were placed on a single small piece of film (2.5 by 3.5 inches). This was accomplished by making the exposures through a slot in a dark slide which could be moved to different positions in front of the photographic material.

The exposed photographic materials were developed for a period of four minutes at a temperature of 65°F in Eastman D–19 developer. The densities were read on a Martens polarization photometer type of visual densitometer.

Results

The results of the experiments are shown graphically in Figs. 4 and 5. The values of density are plotted as a function of the flash times ΔT. In Fig. 4, the curves are given for the three types of materials: Portrait Panchromatic, Panatomic, and Super Speed Portrait films. In

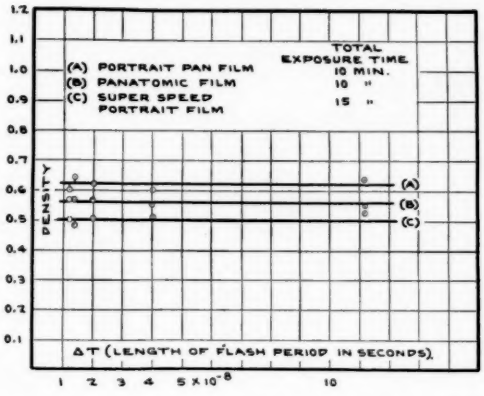
TOTAL EXPOSURE TIME
(A) PORTRAIT PAN FILM 10 MIN.
(B) PANATOMIC FILM 10 "
(C) SUPER SPEED PORTRAIT FILM 15 "
DENSITY
(A)
(B)
(C)
ΔT (LENGTH OF FLASH PERIOD IN SECONDS)
1 2 3 4 5 X 10⁻⁸ 10

FIG. 4. Curves showing density as a function of flash time (constant exposure) for three different photographic materials.

Fig. 5, the results of three sets of exposures for three different (over-all) exposure times are given for Supersensitive Panchromatic film. In Fig. 6, is shown a curve for the Hammer Extra Fast plate, as this plate has been used in similar tests by another worker.[4] All the results obtained in these experiments, as may be seen from these three figures, show no change in photographic effect with changing period of flash. Within the limits of the experimental error, the points indicate in each case a perfectly level density line independent of the length of flash time down to values of 1.2×10^{-8} second. In addition to the results shown in Figs. 4 and 5, it may be added that a large number of similar runs were made in the preliminary tests of the apparatus and there was never found the slightest indication of anything other than a null result.

Discussion

In a recent paper[4] on this subject, L. I. Zimmerman reported results which indicated a new and peculiar type of photographic intermittency effect in the range of flash times covered in the present experiments. These experiments were carried out with a high speed revolving mirror in very much the same manner as those carried out here. For his exposures, Zimmerman employed Hammer Extra Fast Plates as one of his test materials, and these plates were also used as one of the test materials in the present experiments. Briefly, the experimental results reported by

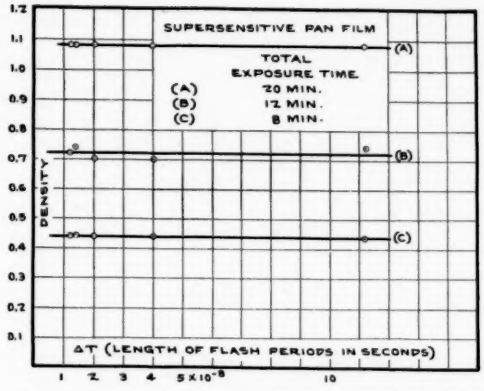

FIG. 5. Curves showing density as a function of flash time (constant exposure) for different total exposure times.

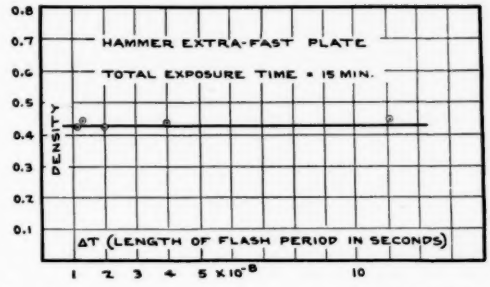

FIG. 6. Curve showing density as a function of flash time (constant exposure) on Hammer Extra Fast plates.

Zimmerman (Fig. 7)* indicated that, when speeds were attained for which the times of the individual flashes were of the order of 7.0×10^{-8} second, the photographic emulsion suddenly dropped enormously in its sensitivity to light. In fact, below this time of flash, it appeared that little or no photographic effect was possible. The results of these experiments were offered in evidence for the hypothesis that a resonance-vibration effect is responsible for the formation of the photographic latent image, the underlying assumption presumably being that, for flash times below a critical minimum value, there is insufficient time for the requisite resonance oscillation to be built up. As may be seen from Figs. 4, 5, 6 and 7, Zimmerman's results are in disagreement with the results obtained here, and they are also in disagreement with what is to be expected from the experimental results obtained by a number of other workers[2] who have measured the photographic intermittency effect at flash times longer than those used by Zimmerman. However, the anomalous effect found by Zimmerman occurs at shorter flash periods than those reached by these other investigators and therefore it cannot be said that his results actually contradict those of the other workers. As mentioned earlier, the results of the other investigators have shown that, under conditions of a sufficiently high frequency of flash (fusion frequency), intermittent exposures of the same average intensity produce identical photographic effects. Now, Zimmerman's experiments were carried out at frequencies which lie entirely in that region above the fusion frequency referred to above. This can be shown by a calculation of the average rate of reception of quanta by a photographic grain for intensities of the order used by him. Zimmerman's results are not in disagreement with the experimental results of other workers at flash times above those for which the anomalous effect was found. The density—ΔT curves in this region (Fig. 7) are practically horizontal, in agreement with the results of the other investigators who worked at the longer flash times. It is, therefore, to be emphasized that Zimmerman's results have indicated an entirely new type of photographic effect, hitherto unknown, and occurring in a new range of flash times. Accordingly, if this effect is a true one, it is to be distinguished from the effect ordinarily referred to under the name "intermittency effect."

Aside from the experimental aspect of his results, the theoretical interpretation which Zimmerman has given his results is considered worthy of some discussion. As previously stated, Zimmerman interprets his results as evidence for a resonance-vibration theory of latent-image formation. This theory assumes that the latent photographic image results from a vibration set up in the photographic grain under the periodically varying force of a light wave. The experimental results, which are characterized by a sharp fall in the photographic action of light at the short flash times, are then considered as evidence for the existence of a critical minimum flash period below which it is impossible to build

* This figure was taken from Zimmerman's paper, J. O. S. A. **23**, 342 (1933).

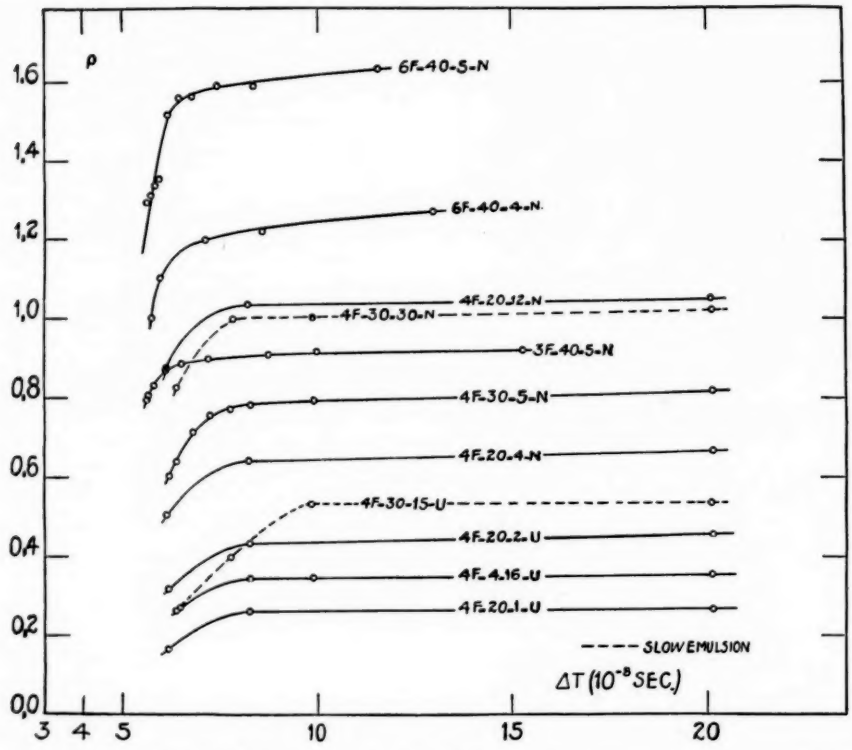

FIG. 7. Curves showing density as a function of flash time for constant exposure (figure taken from paper by L. I. Zimmerman[4]). Nomenclature, e.g., as 4F-20-2-U, designates four-faced mirror-foot-candles of exposing light—total exposure time in minutes—under normal exposure.

up the necessary resonance oscillation in the grain. Leaving aside all discussion of the feasibility of this theory from the photographic standpoint, it is seen that such an interpretation is based entirely on the wave picture of light. It is doubtful, however, whether the wave picture of light is applicable for interpreting experimental results in this case, where a grain is receiving *on the average* one, or less, quanta per flash. In terms of the quantum picture of light, the only possible meaning of this interpretation would appear to be that a fraction of a quantum is being absorbed at the short flash times. Since this contradicts general quantum theoretical prediction (according to which either a whole quantum or nothing is absorbed), considerable doubt is thrown on Zimmerman's results, from a purely theoretical standpoint.

Because of the failure of the present work to confirm Zimmerman's experimental results, and because his results are in apparent conflict with the results which are to be expected from the work of various other investigators, the author is inclined to the opinion that the intermittency effect observed by him is not a true effect but a spurious one introduced by some error in experimental technique. Two possible explanations suggest themselves to account for these results:

(1) The extremely short flash periods in Zimmerman's experiments were obtained by very high rotational speeds (750 to 1000 r.p.s.) of mirrors of square or hexagonal cross section. It is conceivable that the mirror (material of mirror unknown to writer) was distorted sufficiently by centrifugal force at these high speeds to spoil the image of the light source. The danger of this error was lessened in the present experiments by using a very thin mirror made of stellite.

(2) The mirrors used by Zimmerman were mounted in mechanical bearings and it is thought possible that at certain critical speeds, the mirror might have been thrown into vibration, thereby causing the image to oscillate about the position of its mean course across the photographic plate. This is considered much the more likely of the two suggested explanations. In fact, an effect of this type was observed by the author in one experiment at an intermediate speed, and was traced to the vibration of the entire air turbine assembly due to its not being seated firmly on its legs. The cause of the effect was immediately discovered, however, by the elongation of the exposed area on the plate.

Summary

In agreement with former work on the intermittency effect by other investigators and by the author, the results found here indicate that continuous exposures and intermittent exposures for which the frequency of flash is above the fusion frequency, are indistinguishable photographically. By the present experiments, this result has been verified down to flash times of the order of 10^{-8} second.

The results obtained here are not in agreement with results obtained by Zimmerman, who found an anomalous intermittency effect occurring in the region of flash periods of 7×10^{-8} second.

Acknowledgments

It is a pleasure to express my gratitude to Professor J. W. Beams, for the kind loan of his model of a rotor and stator assembly from which a duplicate was made for these experiments, and to my colleague, Mr. V. C. Hall, for putting at my disposal the apparatus for measuring the speed of the rotor, and for his generous help and advice in setting up this apparatus.

First-Order Laws in Asymmetrical Optical Systems

Part I. The Image of a Given Congruence: Fundamental Conceptions*

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(Received June 4, 1936)

INTRODUCTION

IN the September, 1935, issue of this Journal,[1] the author promised to give a theory of asymmetric optical systems paralleling his Gaussian theory, derived from the fundamental invariant, and using systematically his tetrality principle to arrange all theorems in groups of four. Of course, the rays proceeding from a single point in the object space generally do not form a homocentric bundle in the image space. The only fact which we know (and can derive here again) is the law of Malus: that a system of rays which in the object space is normal to a surface (and therefore to a series of parallel surfaces) has this quality still in the image space. To a normal system corresponds a normal system.

I will repeat here what I have said in a previous paper,[2] that such a normal system can be described with the help of a symmetric matrix of two ranks and columns. Therefore, we need not wonder that the laws which govern image formation can be written as formulae between matrices, which, in the special case of Gaussian optics, degenerate into the well-known formulae between numbers. The attempt to generalize from Gaussian optics the conception of angle magnification, and the other conceptions corresponding to it according to the tetrality principle, succeeds in the following way: We consider again a normal congruence which we shall follow through the system. Let us regard the square of the angle magnification as a function of the two congruence parameters. We find that this is a quadratic form which therefore can be described also by a symmetric matrix.

Thus, we find in this paper, for each object congruence, the equation of the image congruence, and its characteristic qualities. For the special cases where point coordinates are allowed, we find the fundamental formulae which were obtained in another way[2] but here we find at the same time all the formulae according to the tetrality principle, which allow us to deal with those cases where the formulae given previously cannot be applied.

A special problem consists in considering the images of the congruences which consist, on object or image side, of rays passing through a single point, and in seeing how the image congruence and its qualities change as the object point is shifted along the principal rays. This leads to the types of systems given by A. Gullstrand, which were investigated.[2] The theory of these system types will be given in part II in a more direct and simpler way than in the previous paper.

I. THE OPTICAL INVARIANT AND THE FUNDAMENTAL RELATIONS BETWEEN THE COEFFICIENTS OF IMAGE FORMATION

We have given an object ray and its corresponding image ray. We choose Cartesian coordinate systems in object and image space, arbitrary except that the z and z' axes coincide with the positive direction of the ray. Then, an arbitrary object ray can be given by the coordinates x and y of the point where it intersects the plane $z=0$, and the direction cosines ξ, η with x and y axis. In the same way, we can give the image ray by x', y', ξ', η'. Of course, x', y', ξ', η' must be univalent functions of x, y, ξ, η and vice versa. We can develop these functions into a Taylor series according to their parameters, and break it off after the linear members, if we restrict ourselves to the actual neighborhood of our principal ray.

According to the tetrality principle, instead of the direction cosines, we shall take as variables the direction cosines multiplied by the refractive indices of the object and image space, n and n', respectively. Then we can write (if we arrange

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* Communication No. 592 from the Kodak Research Laboratories.

[1] Herzberger, J. Opt. Soc. Am. **25**, 295–304 (1935).

[2] Herzberger, Zeits. f. Instrumentenkunde **54**, 329–50, 381–92, 429–41 (1934).

the coefficients in four groups of four)

$$\begin{aligned} x' &= a_{11}x + a_{12}y + b_{11}n\xi + b_{12}n\eta, \\ y' &= a_{21}x + a_{22}y + b_{21}n\xi + b_{22}n\eta, \\ n'\xi' &= c_{11}x + c_{12}y + d_{11}n\xi + d_{12}n\eta, \\ n'\eta' &= c_{21}x + c_{22}y + d_{21}n\xi + d_{22}n\eta. \end{aligned} \qquad \text{(I, 1)}$$

We shall see later that these sixteen coefficients are not independent of each other, but that there are six independent relations between them. Using the so-called matrix calculus, we can give to our equations the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + B\begin{pmatrix} n\xi \\ n\eta \end{pmatrix}, \qquad \text{(I, 2)}$$

$$\begin{pmatrix} n'\xi' \\ n'\eta' \end{pmatrix} = C\begin{pmatrix} x \\ y \end{pmatrix} + D\begin{pmatrix} n\xi \\ n\eta \end{pmatrix},$$

$$\text{with} \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad \text{(I, 3)}$$

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

Eq. (2) can be considered as an abbreviated form of (1). The theory of matrices coordinates to two matrices A and B two certain matrices which it designates as the products, AB and BA, respectively, of the two matrices.

$$AB = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}, & a_{11}b_{12}+a_{12}b_{22} \\ a_{12}b_{11}+a_{22}b_{21}, & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}, \qquad \text{(I, 4)}$$

$$BA = \begin{pmatrix} b_{11}a_{11}+b_{12}a_{21}, & b_{11}a_{12}+b_{12}a_{22} \\ b_{21}a_{11}+b_{22}a_{21}, & b_{21}a_{12}+b_{22}a_{22} \end{pmatrix}.$$

It is obvious that for this matrix multiplication the associative, but not the commutative, law holds true.

$$A \cdot (BC) = (AB) \cdot C. \qquad \text{(I, 5)}$$

The matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = E \qquad \text{(I, 6)}$$

are called the zero and the unit matrix. If $UV = E = VU$, we call V the reciprocal matrix of U and designate it as U^{-1}.

The matrix which results if we interchange ranks and columns is called the transposed matrix. We write

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}. \qquad \text{(I, 8)}$$

The sum of two matrices is the matrix whose members are the sums of the corresponding members of the single matrices.

We can multiply a matrix equation by a matrix from the left or from the right side, but we must always respect the order of the members of the product. This is all we need of the matrix calculus for the present paper.

Now let us go back to Eqs. (1) or (2). If these equations characterize an optical system, our fundamental expression, formula (8),[1] must be invariant. If x_1, y_1, $n\xi_1$, $n\eta_1$, x_1', y_1', $n'\xi_1'$, $n'\eta_1'$, are the coordinates of one arbitrary ray, and x_2, y_2, etc., of another, then we must have

$$\begin{aligned} n'(x_1'\xi_2' - x_2'\xi_1') + n'(y_1'\eta_2' - y_2'\eta_1') \\ = n(x_1\xi_2 - x_2\xi_1) + n(y_1\eta_2 - y_2\eta_1). \end{aligned} \qquad \text{(I, 9)}$$

We have now to substitute (1) in (9), and to consider that the two rays, and therefore their object coordinates, can be arbitrarily chosen; and that therefore the equation obtained is an identity, from which we can find six relations between our coefficients. We find, in matrix notation

$$\tilde{A}C = \tilde{C}A, \quad \tilde{B}D = \tilde{D}B, \quad \tilde{A}D - \tilde{C}B = E. \qquad \text{(I, 10)}$$

These equations can be combined into one, if we apply the rules of matrix multiplication to matrices of matrices,

$$\begin{pmatrix} \tilde{D} & -\tilde{B} \\ -\tilde{C} & \tilde{A} \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}$$

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \tilde{D} & -\tilde{B} \\ -\tilde{C} & \tilde{A} \end{pmatrix}. \qquad \text{(I, 11)}$$

From (6) we can conclude that, besides (10), the equations

$$\begin{aligned} A\tilde{D} - B\tilde{C} = E = D\tilde{A} - C\tilde{B}, \\ A\tilde{B} = B\tilde{A}, \\ C\tilde{D} = D\tilde{C}, \end{aligned} \qquad \text{(I, 12)}$$

hold true, and that therefore we can solve (2) according to the image magnitudes, if we set

$$\begin{pmatrix} x \\ y \end{pmatrix} = \bar{D}\begin{pmatrix} x' \\ y' \end{pmatrix} - \bar{C}\begin{pmatrix} n'\xi' \\ n'\eta' \end{pmatrix},$$
$$\begin{pmatrix} n\xi \\ n\eta \end{pmatrix} = -\bar{B}\begin{pmatrix} x' \\ y' \end{pmatrix} + \bar{A}\begin{pmatrix} n'\xi' \\ n'\eta' \end{pmatrix}. \qquad \text{(I, 13)}$$

II. The Image of a Given Congruence

A congruence is a two-dimensional manifold of rays. We assume that the congruence whose image we are looking for contains the principal ray. We can take as variables either the coordinates x, y, of the point where the ray intersects the plane $z=0$, or the direction cosines $n\xi$, $n\eta$ of the ray. Thus, the congruence rays are given by the equations

$$\begin{aligned} x &= r_{11}n\xi + r_{12}n\eta, \\ y &= r_{21}n\xi + r_{21}n\eta, \end{aligned} \quad \text{or} \quad \begin{pmatrix} x \\ y \end{pmatrix} = R\begin{pmatrix} n\xi \\ n\eta \end{pmatrix} \qquad \text{(II, 1)}$$

$$\text{or} \quad \begin{aligned} n\xi &= p_{11}x + p_{12}y, \\ n\eta &= p_{21}x + p_{22}y, \end{aligned} \quad \text{or} \quad \begin{pmatrix} n\xi \\ n\eta \end{pmatrix} = P\begin{pmatrix} x \\ y \end{pmatrix}.$$

In particular, the normal congruences, that is, the congruences of rays orthogonal to a series of parallel surfaces, are marked by the fact that P and R are symmetric matrices:

$$P \equiv \bar{P}, \quad R \equiv \bar{R}. \qquad \text{(II, 2)}$$

Now let us calculate the image matrix. From (I, 2) we find

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (AR+B)\begin{pmatrix} n\xi \\ n\eta \end{pmatrix} = (A+BP)\begin{pmatrix} x \\ y \end{pmatrix},$$
$$\begin{pmatrix} n'\xi' \\ n'\eta' \end{pmatrix} = (CR+D)\begin{pmatrix} n\xi \\ n\eta \end{pmatrix} = (C+DP)\begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{(II, 3)}$$

$$\text{or} \qquad (AR+B) = R'(CR+D),$$
$$R'CR + R'D - AR - B = 0. \qquad \text{(II, 4)}$$

Multiplication to the right by P, to the left by P', or both, shows that these four equations are valid:

$$\begin{aligned} P'BP + P'A - DP - C &= 0, \\ P'AR + P'B - CR - D &= 0, \\ R'DP + R'C - BP - A &= 0, \\ R'CR + R'D - AR - B &= 0. \end{aligned} \qquad \text{(II, 5)}$$

So we have again four equations simultaneously valid, corresponding to the tetrality principle.

Now, again according to the tetrality principle, there are four distinguished congruences, which are given by

$$\begin{aligned} &(a) \quad P=0, \quad (b) \quad R=0, \\ &(c) \quad P'=0, \quad (d) \quad R'=0. \end{aligned} \qquad \text{(II, 6)}$$

If $P=0$, then $\xi=\eta=0$, which means that the entering bundle is parallel to the principal ray; then the image congruence, which we call the focal congruence, is given, according to (5_1) and (5_3), by

$$\begin{aligned} P_\infty' A &= C, \quad P_\infty' = CA^{-1}, \\ R_\infty' C &= A, \quad R_\infty' = AC^{-1}. \end{aligned} \qquad \text{(II, 7)}$$

If $R=0$, then $x=y=0$, which means that the object bundle consists of all rays through the origin, and the image congruence, according to (5_2) and (5_4), is given by

$$\begin{aligned} P_0' B &= D, \quad P_0' = DB^{-1}, \\ R_0' D &= B, \quad R_0' = BD^{-1}. \end{aligned} \qquad \text{(II, 8)}$$

Analogously, we find that $P'=0$ means that the rays leave the system parallel to the principal ray ($\xi'=\eta'=0$), and the focal congruence in the object space can be found from (5_1) and (5_2):

$$\begin{aligned} DP^\infty &= -C, \quad P^\infty = -D^{-1}C, \\ CR^\infty &= -D, \quad R^\infty = -C^{-1}D, \end{aligned} \qquad \text{(II, 9)}$$

and finally, if we set $R'=0$, we find the object congruence whose image rays pass through the image origin:

$$P^0 = -B^{-1}A, \quad R^0 = -A^{-1}B. \qquad \text{(II, 10)}$$

The relations (I, 10) and (I, 12) show that all these matrices are symmetric matrices. We shall now use these relations to give other forms to our main equations, forms which might be more suitable for further investigation. We multiply (5) by $\bar{B}$, $\bar{A}$, $\bar{D}$, $\bar{C}$, respectively, and get, with the help of (I, 10) and (I, 12),

$$\begin{aligned} P'BP\bar{B} + P'B\bar{A} - DP\bar{B} + E - D\bar{A} &= 0, \\ P'AR\bar{A} + PA\bar{B} - CR\bar{A} - E + C\bar{B} &= 0, \\ R'DP\bar{D} + R'D\bar{C} - BP\bar{D} - E + B\bar{C} &= 0, \\ R'CR\bar{C} + RC\bar{D} - AR\bar{C} + E - A\bar{D} &= 0, \end{aligned} \qquad \text{(II, 11)}$$

or

$$\begin{aligned} (P'B-D)(P\bar{B}+\bar{A}) + E &= 0, \\ (P'A-C)(R\bar{A}+\bar{B}) - E &= 0, \\ (R'D-B)(P\bar{D}+\bar{C}) - E &= 0, \\ (R'C-A)(R\bar{C}+\bar{D}) + E &= 0. \end{aligned} \qquad \text{(II, 12)}$$

Using (II, 7) and (II, 10) we get finally

$$\begin{aligned}(P'-P_0')B(P-P^0)\bar{B}+E=0,\\ (P'-P_\infty')A(R-R^0)\bar{A}-E=0,\\ (R'-R_0')D(P-P^\infty)\bar{D}-E=0,\\ (R'-R_\infty')C(R-R^\infty)\bar{C}+E=0.\end{aligned} \qquad \text{(II, 13)}$$

Characteristic of these equations is the fact that we can calculate for *each* object congruence the image congruence, if we know two symmetric matrices and one asymmetric matrix. This means that we have $2\cdot 3+4=10$ variable coefficients.

III. The Fundamental Conceptions of Optics

I have shown[1] how we can, according to the tetrality principle, coordinate to the fundamental conception of angle magnification γ three other quantities, the projection coefficient δ, and the apparent distance g and g' of object and image origin. Let us see how we can apply these conceptions to the case of an asymmetric system. Our formulae (II, 13) give the image congruence for each object congruence as a whole, but do not give the coordination between object and image ray. From (II, 3) we find

$$\begin{pmatrix}n'\xi\\ n'\eta'\end{pmatrix}=(CR+D)\begin{pmatrix}n\xi\\ n\eta\end{pmatrix},$$
$$\begin{pmatrix}n\xi\\ n\eta\end{pmatrix}=(\bar{C}R'-A)\begin{pmatrix}n'\xi'\\ n'\eta'\end{pmatrix}. \qquad \text{(III, 1)}$$

These two matrices are asymmetric matrices, and of course mutually reciprocal. Either of them would be a very suitable substitute for the usual angle magnification, but this definition would be formal and not very illustrative, and therefore we suggest the following generalization. If we regard the quotient $n'^2(\xi'^2+\eta'^2)/n^2(\xi^2+\eta^2)$ as a function of $n\xi$ and $n\eta$, we get a quadratic function, whose matrix is the symmetric matrix:

$$\Gamma'=(R\bar{C}+\bar{D})(CR+D)=(R-R^\infty)\bar{C}C(R-R^\infty)$$
$$=[C(R'-R_\infty')^2\bar{C}]^{-1}. \qquad \text{(III, 2)}$$

In the same way we get $n^2(\xi^2+\eta^2)/n'^2(\xi'^2+\eta'^2)$ as a quadratic function of $n'\xi'$ and $n'\eta'$, whose matrix is the symmetric matrix

$$\Gamma=(R'C-A)(\bar{C}R'-\bar{A}')=(R'-R_\infty')C\bar{C}(R'-R_\infty')$$
$$=[\bar{C}(R-R^\infty)^2C]^{-1}. \qquad \text{(III, 3)}$$

These two quadratic forms now determine, in a much more illustrative way, everything which is concerned with the angle magnification. If we draw through a fixed point in the object space a parallel to each ray of the object congruence (that is, we draw the spherical image of the congruence), and do the same with the image rays, we see that to each circular cone of object rays ($\xi^2+\eta^2=$ const.) corresponds an elliptical cone of image rays, and *vice versa*. All the cones of the object and image side, respectively, have the same principal sections, and of course the maximum and minimum magnifications on one side are the reciprocals of the same values on the other side.

Verification of these facts might be conceded to the reader. If we call the coefficients of Γ: γ_{11}, γ_{12}, γ_{22}, and the coefficients of Γ': γ_{11}', γ_{12}', γ_{22}', we find the directions of the principal planes by solving the two equations

$$\tan\phi=\eta/\xi,\quad \tan\phi'=\eta'/\xi',$$
and $$\cot 2\phi=(\gamma_{22}-\gamma_{11})/2\gamma_{12}, \qquad \text{(III, 4)}$$
$$\cot 2\phi'=(\gamma_{22}'-\gamma_{11}')/2\gamma_{12}'),$$

respectively.

The extreme magnifications (corresponding to the major and minor axes of the ellipse) are the solutions of the equations

$$u^2-(\gamma_{11}+\gamma_{22})u+(\gamma_{11}\gamma_{22}-\gamma_{12}^2)=0,$$
$$u'^2-(\gamma_{11}'+\gamma_{22}')u'+(\gamma_{11}'\gamma_{22}'-\gamma_{12}'^2)=0. \qquad \text{(III, 5)}$$

The coefficients of (5) are connected by the equations

$$(\gamma_{11}\gamma_{22}-\gamma_{12}^2)(\gamma_{11}'\gamma_{22}'-\gamma_{12}'^2)=1,$$
$$\gamma_{11}+\gamma_{22}=(\gamma_{11}'+\gamma_{22}')(\gamma_{11}'\gamma_{22}'-\gamma_{12}'^2),$$
$$\gamma_{11}'+\gamma_{22}'=(\gamma_{11}+\gamma_{22})(\gamma_{11}\gamma_{22}-\gamma_{12}^2). \qquad \text{(III, 6)}$$

Thus, our two matrices, Γ and Γ', show us all that we want to know about the angle magnification. It may be added here that we have considered not the usual, but the *reduced* angle magnification, found by multiplying all angles by their respective refractive indices. Let us now investigate in the same way the other conceptions corresponding to it, according to the tetrality principle.

In a previous paper[1] we introduced into modern optics one of the main conceptions of the

optics of R. Smith, developed in 1738. If we look at an object through an optical system, the object appears to subtend a different angle than it did without the system, but we can find a distance at which this object will subtend the same angle. This we call the *apparent distance*. This apparent distance is the quotient of the magnitude of the object divided by the angle which it subtends. For instance, if the object is in the image space, we get h'/ξ (using the reduced values, $h'/n\xi$) as a measure of the apparent distance. Of course, for symmetrical systems, this value is the same for all rays of the bundle coming from the same axis point where the eye is situated. But in asymmetric systems, regarding the rays of an arbitrary object congruence, this apparent distance, according to (II, 3), might be represented by the matrix

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (AR+B)\begin{pmatrix} n\xi \\ n\eta \end{pmatrix} \qquad \text{(III, 7)}$$

or the reciprocal value, the *apparent nearness*, by

$$\begin{pmatrix} n\xi \\ n\eta \end{pmatrix} = (\tilde{A}P'-\tilde{C})\begin{pmatrix} x' \\ y' \end{pmatrix}. \qquad \text{(III, 8)}$$

As with the procedure followed for the angle magnification, let us regard, instead of this, the two quadratic functions whose matrices are

$$\begin{aligned} G' &= (R\tilde{A}+\tilde{B})(AR+B) = (R-R^0)\tilde{A}A(R-R^0) \\ &= [\tilde{A}(P'-P_\infty')^2 A]^{-1}, \qquad \text{(III, 9)} \\ H &= (P'A-C)(\tilde{A}P'-\tilde{C}) \\ &= (P'-P_\infty')A\tilde{A}(P'-P_\infty') \\ &= [A(R-R^0)^2\tilde{A}]^{-1}. \end{aligned}$$

Of course, between the matrices G' and H there exist the same relations as between Γ and Γ'.

In the same way, we can consider the apparent distance and nearness of the object plane, given by the equations

$$\begin{pmatrix} x \\ y \end{pmatrix} = (\tilde{A}R'-\tilde{B})\begin{pmatrix} n'\xi' \\ n'\eta' \end{pmatrix}, \qquad \begin{pmatrix} n'\xi' \\ n'\eta' \end{pmatrix} = (C+DP)\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \text{(III, 10)}$$

and determine the symmetric matrices

$$\begin{aligned} G &= (R'D-B)(\tilde{D}R'-\tilde{B}) \\ &= (R'-R_0')D\tilde{D}(R'-R_0') \\ &= [D(P-P^\infty)^2\tilde{D}]^{-1}, \qquad \text{(III, 11)} \\ H' &= (\tilde{C}+P\tilde{D})(C+DP) = (P-P^\infty)\tilde{D}D(P-P^\infty) \\ &= [\tilde{D}(R'-R_0')^2 D]^{-1}. \end{aligned}$$

Finally, we investigate the following conception introduced into optics by Gullstrand. Our congruence meets the coordinate planes $z=0$, $z'=0$, and, in general, gives a univalent projection of the points of the coordinate planes on each other. The ratio of the heights at which the rays meet the coordinate planes is again in Gaussian optics a constant, which is called the projection coefficient. In asymmetric systems we have the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = (A+BP)\begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{pmatrix} x \\ y \end{pmatrix} = (\tilde{D}-\tilde{B}P')\begin{pmatrix} x' \\ y' \end{pmatrix}, \qquad \text{(III, 12)}$$

which leads again to the investigation of the two quadratic forms

$$\begin{aligned} \Delta' &= (P\tilde{B}+\tilde{A})(BP+A) = (P-P^0)\tilde{B}B(P-P^0) \\ &= [\tilde{B}(P'-P_0')^2 B]^{-1}, \\ \Delta &= (P'B-D)(\tilde{B}P'-\tilde{D}) = (P'-P_0')B\tilde{B}(P'-P_0') \\ &= [B(P-P^0)^2\tilde{B}]^{-1}. \end{aligned}$$

Of course, between the coefficients of these pairs of matrices the same relations hold true which we have developed for the tensors of angle magnification.

We see that the four conceptions: angle magnification, apparent distance of the image coordinate plane, apparent nearness of the object plane, and projection coefficient are represented in asymmetric systems with the help of tensors, as are the conceptions which result from these if object and image space are interchanged.

IV. The Connections between These Tensors. The Laws of Huygens and Gauss

The next step, according to our paper on Gaussian optics[1] will be the investigation of our matrices for special congruences. If we set in (III, 2 and 3) $R=0$, $R'=0$, respectively, we find, using the indices from (II, 8),

I	II	III	IV
$(P-P_0')B(P-P^0)\tilde{B}=-E$	$(P'-P_\infty')A(R-R^0)\tilde{A}=E$	$(R'-R_0')D(P-P^\infty)\tilde{D}=E$	$(R'-R_\infty')C(R-R^\infty)\tilde{C}=-E$
$\Delta'=[\tilde{B}(P'-P_0')^2B]^{-1}$	$G'=[\tilde{A}(P'-P_\infty')^2A]^{-1}$	$H'=[\tilde{D}(R'-R_0')^2D]^{-1}$	$\Gamma'=[\tilde{C}(R'-R_\infty')C]^{-1}$
$\Delta=[B(P-P^0)^2\tilde{B}]^{-1}$	$H=[A(R-R^0)^2\tilde{A}]^{-1}$	$G=[D(P-P^\infty)^2\tilde{D}]^{-1}$	$\Gamma=[C(R-R^\infty)^2\tilde{C}]^{-1}$
$G^0=B\tilde{B}$, $G_0'=\tilde{B}B$	$\Gamma^0=A\tilde{A}$, $\Delta_\infty'=\tilde{A}A$	$\Delta^\infty=D\tilde{D}$, $\Gamma_0'=\tilde{D}D$	$H^\infty=C\tilde{C}$, $H_\infty'=\tilde{C}C$
$P_0'=DB^{-1}$, $P^0=-B^{-1}A$	$P_\infty'=CA^{-1}$, $R^0=-A^{-1}B$	$R_0'=BD^{-1}$, $P^\infty=-D^{-1}C$	$R_\infty'=AC^{-1}$, $R^\infty=-C^{-1}D$

$$\Gamma_0'=\tilde{D}D, \quad \Gamma^0=A\tilde{A}. \tag{IV, 1}$$

In the same way we find from (III, 9, 10, 11, 13)

$$\begin{aligned} G_0'&=\tilde{B}B, \quad & G^0&=B\tilde{B}, \\ H^\infty&=C\tilde{C} \quad & H_\infty'&=\tilde{C}C. \end{aligned} \tag{IV, 2}$$

We order these matrices

$$\begin{aligned} A\tilde{A}&=\Gamma^0, \quad & \tilde{A}A&=\Delta_\infty', \\ B\tilde{B}&=G^0, \quad & \tilde{B}B&=G_0', \\ C\tilde{C}&=H^\infty, \quad & \tilde{C}C&=H_\infty', \\ D\tilde{D}&=\Delta^\infty, \quad & \tilde{D}D&=\Gamma_0'. \end{aligned} \tag{IV, 3}$$

Since the matrices which stand opposite each other on the same line have the same characteristic equation, they have therefore the same characteristic roots; and therefore the quadratic forms to which they belong have the same maximum and minimum values.

On the second line, therefore, stands the equivalent to the well-known law of Huygens in Gaussian optics, which says that if we interchange the positions of eye and object in an optical system, the apparent distance of the object from the eye does not change.

In asymmetric systems, the quadratic form G' gives the square of the apparent distance of the image origin from the object origin, and G vice versa.

The third line is related to the law of Gauss concerning the focal lengths of an optical system. For, according to definition, the tensor H_∞' can be considered as the tensor which defines the square of the power on the image side of an optical system, and H^∞, the tensor which defined the square of the power on the object side. They have the same maximum and minimum values.

The first and fourth lines are corresponding laws which have been used in Gaussian optics.[1]

Let us group the formulae found in this paper into four classes:

The first column can always be used if the determinant of B does not vanish, which means, according to (I, 2), that the object and image origin do not correspond to each other. The second column can always be used if the determinant of A does not vanish. In that case, the image origin would be at one of the focal points. The third column can be used if D does not have a vanishing determinant. $|D|=0$ would mean that the object origin was at one of the focal points. The last column can be used if $|C|\neq 0$, in which case one of the focal points would coincide with the infinite point, and the system would be afocal or semi-afocal.

In the next part of this paper we shall investigate the way in which the matrices change if we take special object and image congruences, namely, that all rays pass through one point. For this, we will try to give a complete set of invariants.

We have then to set

$$\begin{aligned} P&=\mu E \quad \text{or} \quad R=(1/\mu)E, \\ P'&=\mu' E \quad \text{or} \quad R'=(1/\mu')E, \end{aligned}$$

respectively, and consider how the matrices of our table change with changes of μ and μ'.

A Verification of the Huygens Wave Surfaces in Uniaxial Crystals

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(Received July 2, 1936)

THE Huygens wave surfaces for doubly refracting uniaxial crystals are discussed in some detail by practically all authors of optical texts, and it appears that the subject of polarization, in general, is best approached from this angle. The purpose of this note is to call attention to a direct simple experimental method for verifying the existence and form of these wave surfaces which was published by Sorby and Stokes[1] in 1877. In a footnote to his paper Stokes remarks, "It seemed pretty certain that some of the phenomena must have been noticed before, though I am not aware that they have been described." After a lapse of almost sixty years it appears that they have not yet been noticed by the writers of optical texts, nor do most physicists seem to have observed them. Except for a brief reference in Gehrcke[2] we have found no account of Sorby's method. Sorby applied to a calcite crystal the usual procedure for determining the index of refraction of a plane parallel slab of material by means of a microscope. An illuminated pinhole[3] is placed in contact with the lower surface of the crystal and the image of the pinhole focused on the cross hairs of a low power microscope.

Case 1. If the crystal faces are cut perpendicular to the optic axis[4] so that the pencil of rays travels along the axis, two positions of the microscope will give distinct images of the source. The separation of these positions is about 15 percent of the thickness of the crystal. Both images are unpolarized and essentially achromatic.

Case 2. If the crystal surfaces contain the optic axis so that the pencil travels perpendicular to the axis, there will be found a point focus and two lines foci, one lying below and the other above the point focus by about 7 percent of the crystal thickness. The lower line focus is parallel to the optic axis and the upper one perpendicular. The point focus is polarized in the principal plane and corresponds to the ordinary wave while the two line foci are polarized perpendicular to the principal plane and are formed from the extraordinary wave. All three are substantially achromatic. These two special cases comprise the directions in which double refraction appears to be absent when judged by the eye. In fact, it is often stated expressly or tacitly assumed that in the first case, where the light travels along the axis, there is no separation into ordinary and extraordinary waves. This is clearly not true if the pencil subtends a finite angle.

Case 3. The general case, as exemplified by that in which the faces of the crystal are cleavage faces, is more complicated. There is now double refraction in the usual sense, i.e., a separation of the images which can be detected by the eye. With the microscope, again three images are formed, but the line foci are displaced laterally from the point focus in the principal plane. The lower line focus lies in the principal plane and the upper one perpendicular to it. When white light is used, both line foci are colored, since the displacement in the principal plane is accompanied by dispersion. The upper line focus in this case has the appearance of a small prismatic spectrum since the dispersion is transverse to the line images. In fact, a reasonably sharp line spectrum is produced with a source such as a mercury arc.

Referring now to the diagrams, Figs. 1 and 2 represent Cases 1 and 2, respectively. The position of the optic axis is indicated by an arrow on each diagram. The pinhole against the lower crystal face acts in each case as a point source S from which two waves spread as in the Huygens construction, and each diagram shows the trace of the two waves OO (ordinary) and EE (extraordinary) in the plane of the particular

[1] H. C. Sorby, Proc. Roy. Soc. **A26**, 384 (1877); G. C. Stokes, ibid., 386 ff.

[2] Gehrcke, Handb. d. Phys. Opt., p. 833.

[3] Sorby actually used a set of ruled lines as object. To bring these into clear focus with the extraordinary wave requires their being parallel or perpendicular to a principal plane of the crystal, in addition to the correct adjustment of the microscope. For this reason a very fine pinhole is preferable, such as may be readily found by inspection of an ordinary silver film on glass.

[4] A single calcite crystal with polished faces in a variety of orientations is available from the Gaertner Scientific Corporation.

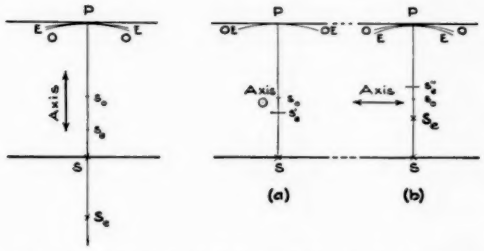

FIG. 1. Wave surfaces in uniaxial crystal cut with faces perpendicular to axis. S is pinhole source and center of ordinary wave OO, S_e is center of curvature of extraordinary wave EE. s_0 and s_e are centers of the waves emerging into air.

FIG. 2. Wave surfaces for crystal cut with axis lying in the faces. Section (a) is perpendicular to axis, section (b) contains the axis. S and s_0 as in Fig. 1. s_e' and s_e'' are the centers of curvature of the principal sections of the ellipsoidal extraordinary wave after emerging.

diagram at the instant of encountering the upper surface at P. It should be noted that the diagrams are not sections of the Huygens wave surfaces since EE and OO, traveling with different velocities, reach P simultaneously and cannot have left S at the same instant. Our interest is restricted in all cases to a narrow conical pencil of rays with PS as axis capable of entering the microscope objective.

When a spherical wave of radius R passes into air through the plane surface of a medium of refractive index N, the radius r of the emergent spherical wave is determined by the equation $Nr=R$. In what follows (as in the diagrams) the large R's will refer to radii of curvature of the incident waves in the crystal at P, the small r's to radii of the waves as they emerge into the air. The subscripts o and e will be used for ordinary and extraordinary waves, respectively. Thus the two refractive indices are denoted by N_0 and N_e. It will be convenient to denote the ratio N_0/N_e by the symbol γ, which is necessarily equal also to the ratio of the major to the minor axis of the Huygens extraordinary wave ellipsoid.

The quantities which are easily accessible to direct measurement are R_0, r_0, and r_e. R_0, the radius of the incident ordinary wave at P, is in all cases equal to the crystal thickness PS. The emergent radii r are measured by focusing the microscope on the upper crystal face, then bringing the emergent wave to a sharp point or line focus, and measuring the distance Ps through which the microscope was moved. It will be seen that the large S's designate the apparent sources (centers of curvature) of the incident waves and the small s's those of the emergent waves, so that the distances PS, Ps are equal, respectively, to R, r.

Case 1, Fig. 1. Since PS coincides with the optic axis the entire figure is one of revolution about PS. OO is the trace of a sphere of radius R_0 equal to PS, the crystal thickness. EE is a section of an ellipsoid of revolution with minor semiaxis PS. In the neighborhood of P it is equivalent to a sphere of radius R_e or PS_e. The radius of curvature of an ellipsoid at a point on the minor axis is $\gamma^2 B$ where A and B are the major and minor semiaxes and $\gamma=A/B$. Hence $R_e=\gamma^2 R_0$. Both waves remain spherical on emergence from the crystal and may be brought to point foci by the microscope. Since the two waves in the crystal travel with equal speeds along PS, the ordinary refractive index must be used in computing the emergent radius in each case, so that we may write

$$r_0=R_0/N_0, \quad r_e=\gamma^2 R_0/N_0, \quad r_e=\gamma^2 r_0.$$

Case 2, Figs. 2a and 2b. The diagram and calculation for OO, R_0 and r_0 are precisely the same as in Case 1. Both incident and emergent extraordinary waves, however, are ellipsoidal with two principal radii of curvature, so that the microscope cannot bring this wave to a point focus. Fig. 2a shows a section at right angles to the optic axis, and it is clear that in this plane the traces 00, and EE, coincide. But the velocities of the two waves are different, so that the microscope will discover, below the apparent point image at s_0 ($PS_0=r_0=R_0/N_0$) a line image at s_e' such that $Ps_e'=r_e'=R_0/N_e$. This line source is evidently parallel to the optic axis. From the equations for r_0 and r_e' it is evident that $r_e'/r_0=\gamma$.

Fig. 2b represents a section parallel to the axis. Here the trace EE is elliptical and of sharper curvature than OO since it has R_0 as its major semi-axis. The radius of curvature in the neighborhood of P is $R_e''=R_0/\gamma^2$, and the radius after emergence into the air is $r_e''=R_e''/N_e=R_0/\gamma N_0$ so that $r_0/r_e''=\gamma$. Evidently the microscope will find this trace as a line image s_e'' perpendicular to the optic axis and lying above the point image s_0.

The third or general case is more complicated and need not be treated here. For this and for the extension to cover biaxial crystals see Stokes' paper. A set of measurements and calculations by the present writers using a sodium arc and a calcite crystal will illustrate the quantitative use of the method.

Case 1.	Measured	Calculated
	$R_0 - 2.122$	$\gamma = (r_e/r_0)^{\frac{1}{2}} = 1.117$
	$r_0 - 1.278$	$N_0 = R_0/r_0 = 1.660$
	$r_e - 1.594$	$N_e = N_e/r = 1.486$
Case 2.	Measured	Calculated
	$R_0 - 2.466$	$\gamma = r_e'/r_0 = 1.113$
	$r_0 - 1.485$	$\gamma = r_0/r_e'' = 1.116$
	$r_e' - 1.653$	$N_0 = R_0/r_0 = 1.661$
	$r_e'' - 1.330$	$N_e = N_0/\gamma = 1.490$

Average values	$N_0 - 1.660$	$N_e - 1.488$	$\gamma - 1.115$
I.C.T. values	1.6583	1.4864	1.1156

An intriguing feature of the above calculation for Case 1 is that the two indices may be determined from measurements made on a pencil of rays which travel along the axis, a result directly contrary to the usual loose assertion that no difference of any kind exists in this direction. Furthermore it is rather entertaining that each of these pencils which result from double refraction along the axis is unpolarized.

In conclusion it may be said that the value of Sorby's method lies not in the precision of the indices for these may be found more accurately by other methods but rather in the verification quantitatively of Huygens wave surfaces in a uniaxial crystal. It is admitted that the above material is not new in the literature but it is the feeling of the authors that it is unknown to many teachers of optics. Its simplicity and cogency should warrant its republication.

The Measurement of Small Retardations with the Polarizing Microscope

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(Received June 15, 1936)

A modification of the Köhler method is described which facilitates the measurement of the small retardations frequently found in minute biological objects.

IN order to make full use of the polarizing microscope for the study of the fine structure of biological materials it is desirable that quantitative methods be utilized. Fortunately instruments for this purpose are available, and it is only necessary to choose the best methods of employing them. One of the chief difficulties encountered in such work arises from the requirement that very small phase differences be measurable, since the biological objects of interest are often only weakly birefringent or of very small size. Perhaps the best compensator for such purposes is that of the rotating mica plate of small retardation after Köhler.[1]

Even when a suitable compensator is applied circumstances of a sort peculiar to the material are apt to lead to considerable difficulty in obtaining sufficient accuracy. The investigator rarely deals with an object which covers the field of view entirely, hence during adjustment of the compensator not only does the object change in apparent brightness but the background varies also. These conflicting changes are confusing to the eye and may lead to serious error. We have overcome this difficulty to a considerable extent by the following modification of Köhler's method.

The intensity I of light emerging from an optical system containing two polarizing prisms and two birefringent objects placed between them can be found from the equation:

$$\begin{aligned} I = I_0\{&\cos^2\gamma \\ &+\sin 2\alpha \sin 2(\beta-\alpha)\cos 2(\gamma-\beta)\sin^2 m\pi \\ &+\cos 2\alpha \sin 2(\beta-\alpha)\sin 2(\gamma-\beta)\sin^2 m'\pi \\ &+\sin 2\alpha \cos^2(\beta-\alpha)\sin 2(\gamma-\beta)\sin^2(m'+m)\pi \\ &-\sin 2\alpha \sin^2(\beta-\alpha)\sin 2(\gamma-\beta)\sin^2(m'-m)\pi\}, \end{aligned} \quad (1)$$

where I_0 is the initial intensity of the plane polarized beam transmitted by the polarizer; α, β, and γ are the angles between the vibration plane of the polarizer and the principal axis of the first object, the same of the second object, and the plane of vibrations passed by the analyzer, respectively; and m and m' are the respective relative retardations of the first and second objects in fractions of a wave-length.[2] For crossed Nicol prisms $\gamma=\pi/2$, hence

$$\begin{aligned} I = I_0\{&\cos 2\alpha \sin 2(\beta-\alpha)\sin 2\beta \sin^2 m'\pi \\ &-\sin 2\alpha \sin 2(\beta-\alpha)\cos 2\beta \sin^2 m\pi \\ &+\sin 2\alpha \cos^2(\beta-\alpha)\sin 2\beta \sin^2(m'+m)\pi \\ &-\sin 2\alpha \sin^2(\beta-\alpha)\sin 2\beta \sin^2(m'-m)\pi\}. \end{aligned} \quad (2)$$

If m is the phase difference of the unknown object and m' that of the compensating plate, the latter covering the entire field, then Eq. (2) becomes upon letting $m=0$:

$$I_F = I_0 \sin^2 m'\pi \sin^2 2\beta, \quad (3)$$

which applies to the part of the field in which no object of unknown retardation appears. With the object in diagonal position $\alpha=\pi/4$, and thus for the appearance of such objects we have from Eq. (2):

$$\begin{aligned} I_X = \tfrac{1}{2} I_0\{&\sin^2 2\beta[\sin^2(m'+m)\pi+\sin^2(m'-m)\pi] \\ &+\sin 2\beta[\sin^2(m'+m)\pi-\sin^2(m'-m)\pi] \\ &+2\cos^2 2\beta \sin^2 m\pi\}. \end{aligned} \quad (4)$$

The usual Köhler formula is obtained by letting $I_X=0$, and then to the approximation that $\sin(m'+m)\pi$, $\sin(m'-m)\pi$, and $\sin m\pi$ are equal to their arguments it is found that

$$m = -m' \sin 2\beta_1. \quad (5)$$

It is a more accurate procedure in the case of small objects which do not entirely cover the field of view to adjust the compensator to the point at which field and object are of equal in-

[1] A. Köhler, Zeits. f. wiss. Mikroskopie **38**, 29 (1921).

[2] For derivation see *Handbuch der Physik*, vol. XX, p. 762.

tensity. Letting $I_F = I_X$ we have accordingly, to the same approximation used in obtaining Eq. (5):

$$m = -2m' \sin 2\beta_2. \tag{6}$$

The subscripts on the β's of Eqs. (5) and (6) indicate that they are particular values for the compensation methods employed. The use of the second method (Eq. (6)) is favored by the fact that for positions of the compensating plate in the range of β about equal to β_2 both object and field are changing in intensity in opposite directions as the mica plate is rotated, hence the rapid variations in the relative intensities of object and field make the setting of the compensator to equal intensity for these two regions particularly easy. Since sin 2β can not be greater than unity in absolute value, the maximum m measurable according to Eq. (5) is equal to m', while in (6) it is $2m'$. Thus the instrument has two ranges of applicability, and in spite of the fact that the sensitivity would appear to be less in the second case because of the larger range, in practice we have found that the increased accuracy more than overbalances this defect. The two methods, however, serve as valuable checks upon each other.